Week 11
Nov 15

Lectures

1. Ⓘ Green's Theorem
2. Ⓘⅈ Double $\iint$ vs line integral
3. Ⓘⅈⅈ Area formula
4. Ⓘⅴ Simply-connected region
5. Ⓥ Divergence and Circulation density

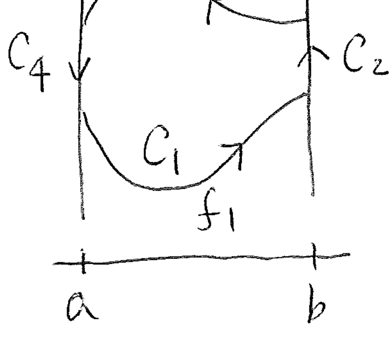
Ⓘ Green's Theorem

<u>Theorem 1</u> Let $\vec{F} = M\hat{i} + N\hat{j}$ be a smooth v.f. on an region D which is bounded by a single closed, curve C. Then

$$\iint_D (N_x - M_y)\, dA = \oint_C M\,dx + N\,dy,$$ where C is oriented in the anticlockwise direction.

Pf. Assume D is a type I and type II region.

$$D = \{(x,y) : f_1(x) \le y \le f_2(x),\ a \le x \le b\} \text{ type I}$$



$C = C_1 + C_2 + C_3 + C_4$.

C_1 $\vec{r}(x) = x\hat{i} + f_1(x)\hat{j}$, $x \in [a,b]$,

C_2 $\vec{r}(y) = b\hat{i} + y\hat{j}$, $y \in [f_1(b), f_2(b)]$,

$-C_3$ $\vec{r}(x) = x\hat{i} + f_2(x)\hat{j}$, $x \in [a,b]$

$-C_4$ $\vec{r}(y) = a\hat{i} + y\hat{j}$, $y \in [f_1(a), f_2(a)]$.

Claim

$$\oint_C M\,dx = -\iint_D M_y\,dA \qquad \text{— (1)}$$

$$\oint_C M\,dx = \left(\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}\right) M\,dx,$$

$$\int_{C_1} M\,dx = \int_a^b M(x, f_1(x))\frac{dx}{dx}\,dx = \int_a^b M(x, f_1(x))\,dx.$$

$$\int_{C_2} M\,dx = \int_{f_1(b)}^{f_2(b)} M(b,y)\frac{db}{dy}\,dy = \int_{f_1(b)}^{f_2(b)} M(b,y)\times 0\times dy = 0$$

$$\int_{C_3} M\,dx = -\int_{-C_3} M\,dx = -\int_a^b M(x, f_2(x))\frac{dx}{dx}\,dx$$

$$= -\int_a^b M(x, f_2(x))\,dx.$$

$$\int_{C_4} M\,dx = -\int_{-C_4} M\,dx = -\int_{f_1(a)}^{f_2(a)} M(a,y)\frac{da}{dy}\,dy = \int_{f_1(a)}^{f_2(a)} M(a,y)\times 0\times dy = 0$$

$$\therefore \text{LHS} = \int_a^b M(x, f_1(x)) - M(x, f_2(x))\,dx. \qquad (2)$$

$$\iint_D M_y\,dA = \int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}(x,y)\,dy\,dx$$

$$= \int_a^b M(x,y)\Big|_{f_1(x)}^{f_2(x)}\,dx$$

$$= \int_a^b M(x, f_2(x)) - M(x, f_1(x))\, dx \quad \text{— ③}$$

By comparing ② and ③, ① holds.

Similarly, when D is of Type II,

$$D = \{ (x, y) : \begin{matrix} g_1(y) \le x \le g_2(y), \\ c \le y \le d \end{matrix} \}$$

We can show

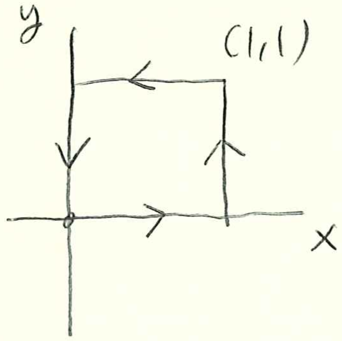
$$\oint_C N\, dy = \iint_D N_x\, dA \quad \text{— ④}$$

Adding up ① and ④, we get Green's Theorem. #

II Double $\iint$ vs Line $\int$.

Green's Theorem relates double integral to line integral. As application, we convert the evaluation of line integral to double integral, and vice versa.

e.g. Evaluate $\oint_C xy\, dy - y^2 dx$ where C is the boundary of the square R in anticlockwise way.



To avoid to do line integral 4 times, we use Green's thm,

$$\oint_C xy\, dy - y^2 dx = \iint_R \frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(-y^2)\, dA$$

$$= 3\iint_R y\,dA$$

$$= 3\int_0^1\int_0^1 y\,dy\,dx$$

$$= \frac{3}{2}. \quad \#$$

(III) Area formula.

<u>Theorem 2</u> Let D be a region enclosed by the closed curve C. then the area of D

$$|D| = \frac{1}{2}\oint_C x\,dy - y\,dx$$

$$= \oint_C x\,dy$$

$$= -\oint_C y\,dx.$$

Pf: Select $\vec{F} = -y\hat{i}$. then $N_x - M_y = 1$. Green's thm,

$$\oint_C -y\,dx = \iint_D 1\,dA = |D|.$$

Select $\vec{F} = x\hat{j}$. then $N_x - M_y = 1$, so

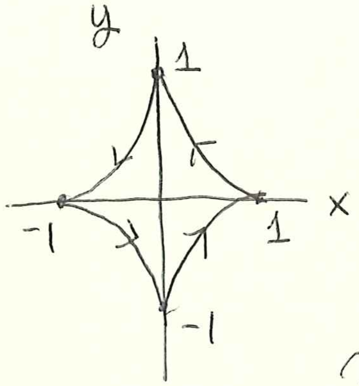
$$\oint x\,dy = \iint_D 1\,dA = |D|$$

Select $\vec{F} = \frac{1}{2}(-y\hat{i} + x\hat{j})$. then $N_x - M_y = 1$, so

$$\frac{1}{2}\oint(-y\,dx + x\,dy) = |D|.$$

e.g. Find the enclosed area of the astroid

$$x(t) = \cos^3 t, \quad y(t) = \sin^3 t, \quad t \in [0, 2\pi]$$



$$\vec{r}(t) = \cos^3 t\,\hat{i} + \sin^3 t\,\hat{j}$$

$$\vec{r}\,'(t) = -3\cos^2 t \sin t\,\hat{i} + 3\sin^2 t \cos t\,\hat{j}.$$

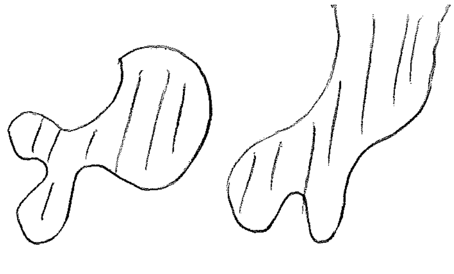
Use area formula,

$$\begin{aligned} x\,dy - y\,dx &= \cos^3 t \times 3\sin^2 t \cos t - \sin^3 t \times (-3\cos^2 t \sin t)\ dt \\ &= 3\cos^2 t \sin^2 t\,(\cos^2 t + \sin^2 t) \\ &= 3\cos^2 t \sin^2 t \\ &= \frac{3}{4}\sin^2 2t. \end{aligned}$$

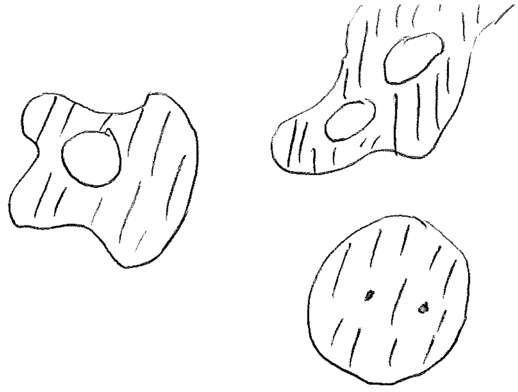
$$\begin{aligned} \therefore \text{area} &= \frac{1}{2} \times \oint_C x\,dy - y\,dx \\ &= \frac{1}{2} \times \frac{3}{4} \int_0^{2\pi} \sin^2 2t\,dt \\ &= \frac{1}{2} \times \frac{3}{4} \times \frac{1}{2} \int_0^{2\pi} (1 + \cos 4t)\,dt \\ &= \frac{1}{2} \times \frac{3}{4} \times \frac{1}{2} \times 2\pi \\ &= \frac{3}{8}\pi. \ \# \end{aligned}$$

(IV) Simply-Connected Region

An open region is called <u>simply-connected</u> if it has no holes or punctured.



simply-connected regions



multi-connected regions

<u>Theorem</u> 3 Let $\vec{F} = M\hat{i} + N\hat{j}$ be a smooth v.f. on a simply-connected region G. Then $\vec{F}$ is conservative if and only if the component test is fulfilled:

$$M_y = N_x .$$

Pf. $\Rightarrow$ done already. Recall if $\vec{F} = \nabla g$, ie,

$$\frac{\partial g}{\partial x} = M, \quad \frac{\partial g}{\partial y} = N. \quad \text{then} \quad M_y = g_{xy} = g_{yx} = N_x .$$

$\Leftarrow$) Let C be any simply, closed curve lying in G.

The

$$\oint_C M\,dx + N\,dy = \iint_D N_x - M_y \, dA$$

$$= 0$$

$\therefore$ $\vec{F}$ is indept of path $\Leftrightarrow$ $\vec{F}$ is conservative. #

We summarize what known about conservative v.f.

~ $\vec{F}$ is conservative $\Longleftrightarrow$ $\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$ any paths C_1, C_2 with same endpts

$$\Longleftrightarrow \oint_C \vec{F}\cdot d\vec{r} = 0 .$$

~ $\vec{F}$ is conservative $\Longrightarrow$ Component Test $N_x = M_y$.

~ $\vec{F}$ defined on a simply-connected region. Then

Component Test $\Longrightarrow$ $\vec{F}$ conservative.

~ $\vec{F} = \dfrac{-y}{x^2+y^2}\hat{i} + \dfrac{x}{x^2+y^2}\hat{j}$ defined on the punctured $\mathbb{R}^2$.

Component Test satisfied but

$$\oint_C \vec{F}\cdot d\vec{r} = 2\pi \quad (C \text{ the unit circle})$$

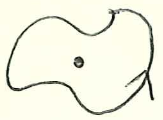
So $\vec{F}$ is not conservative in $\mathbb{R}^2 \setminus (0,0)$.

(V) Circulation density and Divergence.

Green's Theorem helps to "localize" circulation and flux.

Take a closed curve C around a point $(x,y) = x\hat{i} + y\hat{j}$

C



For a v.f $\vec{F}$ on open region G,

$$\oint_C M\,dx + N\,dy = \iint_D (N_x - M_y)\,dA,$$

where D is enclosed by C. So

$$(N_x - M_y)(x,y) = \lim_{d\to 0} \frac{1}{|D|}\oint_C M\,dx + N\,dy$$

as the diameter d of D tends to 0. This suggests to define the circulation density of $\vec{F}$ at (x,y) to be

$$(N_x - M_y)(x,y).$$

On the other hand, the flux of $\vec{F}$ across C is

$$\oint_C -N\,dx + M\,dy = \oint_C \vec{F}\cdot\hat{n}\,ds$$

$$= \iint_D M_x + N_y \, dA.$$

So,
$$(M_x+N_y)(x,y) = \lim_{d\to 0} \frac{1}{|D|} \oint_C -N\,dx + M\,dy.$$

- This suggests to define the flux density or the divergence of $\vec{F}$ at (x,y) to be

$$(M_x+N_y)(x,y).$$

Surfaces.

(I) p-surfaces and surfaces

(II) Examples.

The discussion is somewhat parallel to p-curves and curves.

A <u>parametric surface</u> is a continuous map from a region in $\mathbb{R}^2$ to $\mathbb{R}^3$:

$$\vec{r}(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k},$$

$(u,v) \in D$. It is <u>smooth</u> if x, y, z are continuously differentiable in (u,v). It is <u>regular</u> if $\vec{r}_u, \vec{r}_v$ are linearly independent, or equivalently,

$$|\vec{r}_u \times \vec{r}_v| > 0 \quad \text{on } D.$$

A <u>surface</u> S is a subset in $\mathbb{R}^3$ that admits a parametrization, that is, $\exists$ a parametric surface $\vec{r}$ such that $\vec{r}(D) = S$. Usually we assume $\vec{r}$ is smooth and regular, and 1-1 in the interior of D.

(II) Examples.

e.g. the sphere $S = \{(x,y,z) : x^2+y^2+z^2 = r^2\}$

A standard parametrization is

$$\vec{r}(\varphi,\theta) = (r\sin\varphi\cos\theta,\ r\sin\varphi\sin\theta,\ r\cos\varphi)$$

$(\varphi,\theta) \in [0,\pi]\times[0,2\pi]$.

$\vec{r}$ is smooth and regular :

$$\vec{r}_\varphi = (r\cos\varphi\cos\theta,\ r\cos\varphi\sin\theta,\ -r\sin\varphi)$$

$$\vec{r}_\theta = (-r\sin\varphi\sin\theta,\ r\sin\varphi\cos\theta,\ 0)$$

$$\vec{r}_\varphi \times \vec{r}_\theta = \begin{vmatrix} i & j & k \\ & \vec{r}_\varphi & \\ & \vec{r}_\theta & \end{vmatrix} = \cdots .$$

$$|\vec{r}_\varphi \times \vec{r}_\theta| = |r^2\sin^2\varphi\cos\theta\,\hat{i} + r^2\sin^2\varphi\sin\theta\,\hat{j} + r^2\sin\varphi\cos\theta\,\hat{k}|$$

$$= r^2\sin\varphi$$

$$> 0 \quad \text{in } (0,\pi)\times[0,2\pi].$$

It is also 1-1 on $(0,\pi)\times(0,2\pi)$.

e.g. the cylinder $\{(x,y,z) : (x-a)^2+y^2 = R^2,\ z\in\mathbb{R}\}$

The circle $(x-a)^2+y^2=R^2$ can be parametrized by

$$x-a = R\cos\theta$$

$$y = R\sin\theta, \quad \theta\in[0,2\pi]$$

Hence $\vec{r}(\theta,z) = (a+R\cos\theta)\,\hat{i} + R\sin\theta\,\hat{j} + z\,\hat{k}$

gives a parametrization of the cylinder.

$$\vec{r}_\theta = -R\sin\theta\,\hat{i} + R\cos\theta\,\hat{j} + 0\,\hat{k}$$
$$\vec{r}_z = 0\,\hat{i} + 0\,\hat{j} + \hat{k}$$

$$|\vec{r}_\theta \times \vec{r}_z| = |R\cos\theta\,\hat{i} + R\sin\theta\,\hat{j} + 0\,\hat{k}|$$
$$= R > 0 \quad \therefore \text{ regular}.$$

e.g. the graph case

$$S = \{(x, y, z) : (x, y) \in D,\ z = f(x, y)\}$$

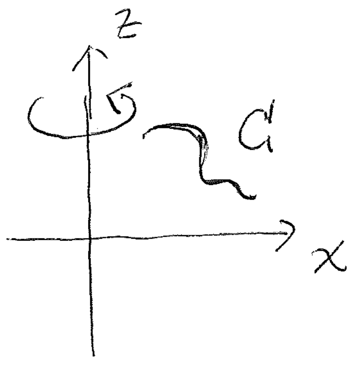
Just use (x, y) to parametrize S.

$$\vec{r}(x, y) = x\,\hat{i} + y\,\hat{j} + f(x, y)\,\hat{k}$$
$$\vec{r}_x = \hat{i} + 0\,\hat{j} + f_x\,\hat{k}$$
$$\vec{r}_y = 0\,\hat{i} + \hat{j} + f_y\,\hat{k}$$

$$|\vec{r}_x \times \vec{r}_y| = (1 + f_x^2 + f_y^2)^{\frac{1}{2}} > 0, \text{ always regular.}$$

e.g. Surfaces of Revolution

Let C $(x(t), z(t))$ be a curve in xz-plane. Revolve it around the z-axis to get a surface S.



the standard parametrization:

$$\vec{r}(\theta, t) = x(t)\cos\theta\,\hat{i} + x(t)\sin\theta\,\hat{j} + z(t)\,\hat{k}.$$

$$\vec{r}_\theta = -x(t)\sin\theta\,\hat{i} + x(t)\cos\theta\,\hat{j} + 0\,\hat{k}$$
$$\vec{r}_t = x'(t)\cos\theta\,\hat{i} + x'(t)\sin\theta\,\hat{j} + z'(t)\,\hat{k}$$

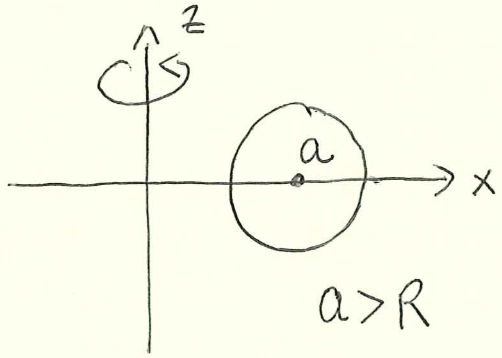
$$\vec{r}_\theta \times \vec{r}_t = x(t) z'(t) \cos\theta \,\hat{i} + x(t) z'(t) \sin\theta \,\hat{j} - x(t) x'(t) \hat{k}$$

$$|\vec{r}_\theta \times \vec{r}_t| = |x(t)| \left(x'^2(t) + y'^2(t) \right)^{\frac{1}{2}}.$$

Hence regular if C is a regular curve and $|x(t)| > 0$.

A useful example is the torus obtained by rotating a circle

$$(x-a)^2 + z^2 = R^2$$



Its standard parametrization is

$$\vec{r}(\alpha, \theta) = (a + R\cos\theta) \cos\alpha \,\hat{i} + R\sin\theta \sin\alpha \,\hat{j} + R\sin\theta \,\hat{k}.$$

(We have changed some notation.)